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Some new and short proofs for a class of Caffarelli–Kohn–Nirenberg type inequalities

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Abstract

In this note we provide simple and short proofs for a class of inequalities of Caffarelli–Kohn–Nirenberg type with sharp constants. Our approach suggests some definitions of weighted Sobolev spaces and their embedding into weighted L^2 spaces. These may be useful in studying solvability of problems involving new singular PDEs.

Keywords: Caffarelli–Kohn–Nirenberg inequality; Hardy inequality; Sharp constants; Singular PDEs

1. Introduction

As we all know, the elementary result "*There are no real solutions of the quadratic equation* $At^2 + Bt + C = 0$ *if and only if the discriminant* $B^2 - 4AC < 0$" has nontrivial consequences in analysis. As an example, we recall that an easy proof of the Cauchy–Schwarz inequality

$$\left|\langle u, v\rangle\right| \leqslant \|u\|\|v\| \quad \forall u, v \in H \tag{1.1}$$

(say, for $u, v \neq 0$, without loss of generality) in a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$ simply involves the fact that the quadratic inequality

$$\|u - tv\|^2 \geqslant 0 \quad \forall t \in \mathbb{R}$$

is equivalent to its discriminant being negative or zero, the latter case yielding equality in (1.1) and in above (with $t = \langle u, v\rangle/\|v\|^2$).

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I was recently reminded of this elementary (and natural) idea when, during a conference, Esteban used it to derive a short proof of Hardy's inequality and of other inequalities involving Schrödinger and Dirac operators (cf. [6]).

In this paper we use a similar approach to provide short and elementary proofs for a class of Caffarelli–Kohn–Nirenberg inequalitites (CKN). In addition, we explicitly determine corresponding sharp constants. Our approach further suggests definitions of some weighted Sobolev spaces and their continuous embedding into weighted L^2 spaces together with best embedding constants. We remark that a different simple proof of (CKN) using a scaling argument can be found in [8] (see also Theorem 3.1 in [1] and Corollary 2.3(ii) below).

2. A class of weighted inequalities

In [2] Caffarelli, Kohn and Nirenberg proved a rather general interpolation inequality with weights. As the authors mentioned, their proof although elementary was however rather long. In fact, we point out that some other inequalities from [3,7,9,10] (which turned out to be special cases of their general inequality) were also used in their proof. In this note, the only tools we use are integration by parts and the elementary algebraic fact mentioned in the beginning.

Let us recall the L^2 version of the Caffarelli–Kohn–Nirenberg inequality [2]:

$$\hat{C} \int\limits_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2c}}\, dx \leqslant \left(\int\limits_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}\, dx \right)^{1/2} \left(\int\limits_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\, dx \right)^{1/2} \tag{2.1}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$, where $a, b, c < \frac{N}{2}$, $c = \frac{1}{2}(a + b + 1)$ and $\hat{C} := \hat{C}(a, b)$ is a suitable positive constant. Therefore, if $D_\alpha^{1,2}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D_\alpha^{1,2}} := \left(\int\limits_{\mathbb{R}^N} |x|^{-2\alpha} |\nabla u|^2\, dx \right)^{1/2} \tag{2.2}$$

and $L_\alpha^2(\mathbb{R}^N)$ denotes the weighted Lebesgue space with norm

$$\|u\|_{L_\alpha^2} := \left(\int\limits_{\mathbb{R}^N} |x|^{-2\alpha} |u|^2\, dx \right)^{1/2}, \tag{2.3}$$

then (2.1) implies that, with the parameters a, b and c given as above, the weighted Sobolev type space $E := D_a^{1,2}(\mathbb{R}^N) \cap L_b^2(\mathbb{R}^N)$ is continuously embedded in $L_c^2(\mathbb{R}^N)$. Moreover, in the present situation, it is known [5] that the best constant $\hat{C} := \hat{C}(a + 1, a)$ in (2.1) is never attained (see also [4,11] for other cases). If $\alpha = 0$ in (2.2) (or in (2.3)) we simply write $D_0^{1,2}(\mathbb{R}^N) = D^{1,2}(\mathbb{R}^N)$ (or $L_0^2(\mathbb{R}^N) = L^2(\mathbb{R}^N)$).

When $a = 1$ and $b = 0$ (so that $c = 1$), note that (2.1) reduces to Hardy inequality, which yields the continuous embedding $D^{1,2}(\mathbb{R}^N) \subset L_1^2(\mathbb{R}^N)$. We now provide our simple proof of a version of the Caffarelli–Kohn–Nirenberg inequality (2.1) without any other restriction on the parameters $a, b, c \in \mathbb{R}$ besides $c = \frac{1}{2}(a + b + 1)$. We also exhibit the corresponding sharp constants. We note that the function u in the theorem below is assumed to be real-valued, but minor modifications show the result to hold for complex-valued u as well.

Theorem 2.1 *(Caffarelli–Kohn–Nirenberg). For all $a, b \in \mathbb{R}$ and $u \in C_0^\infty(\mathbb{R}^N \backslash \{0\})$ one has the inequality*

$$\hat{C} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}\, dx \leqslant \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}\, dx \right)^{1/2} \left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\, dx \right)^{1/2}, \tag{2.4}$$

where the constant $\hat{C} = \hat{C}(a, b) := \frac{|N-(a+b+1)|}{2}$ is sharp.

Proof. Given $u \in C_0^\infty(\mathbb{R}^N \backslash \{0\})$ arbitrary and $a, b \in \mathbb{R}$, we have

$$\int_{\mathbb{R}^N} \left| \frac{\nabla u}{|x|^b} + t \frac{x}{|x|^{a+1}} u \right|^2 dx \geqslant 0 \tag{2.5}$$

for every $t \in \mathbb{R}$. In other words,

$$\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\, dx + t^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}\, dx + 2t \int_{\mathbb{R}^N} u \frac{x}{|x|^{a+b+1}} \cdot \nabla u\, dx \geqslant 0 \tag{2.6}$$

for every $t \in \mathbb{R}$. Denote the last integral by $[I]$. Through integration by parts,

$$[I] = -[I] - \big[N - (a+b+1)\big] \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}\, dx,$$

hence

$$[I] = -\frac{N - (a+b+1)}{2} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}\, dx.$$

Therefore, (2.6) reads

$$At^2 - Bt + C \geqslant 0$$

for every $t \in \mathbb{R}$, where

$$A = \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}\, dx, \qquad B = \big[N - (a+b+1)\big] \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}\, dx, \qquad C = \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\, dx.$$

This is equivalent to $B^2 - 4AC \leqslant 0$, or

$$\big[N - (a+b+1)\big]^2 \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}\, dx \right)^2 \leqslant 4 \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}\, dx \right) \left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\, dx \right).$$

The proof is complete. It is clear that the above approach yields sharp constants. $\square$

Remark 2.2. We should note the symmetry of the Caffarelli–Kohn–Nirenberg inequality as written in (2.4) as well as the arbitrariness of the real parameters a, b (as we took our test functions supported away from the origin). Let us therefore denote by $H^1_{a,b}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N \backslash \{0\})$ with respect to the weighted Sobolev norm

$$\|u\|_{H^1_{a,b}} := \left(\int_{\mathbb{R}^N} \left[\frac{|u|^2}{|x|^{2a}} + \frac{|\nabla u|^2}{|x|^{2b}} \right] dx \right)^{1/2},$$

and by $L^2_c(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N\backslash\{0\})$ with respect to the weighted Lebesgue norm mentioned in the beginning of this section. Then (2.4) implies that, for $a+b+1\neq N$, [1] we have the continuous embedding

$$H^1_{a,b}\big(\mathbb{R}^N\big)\subset L^2_{(a+b+1)/2}\big(\mathbb{R}^N\big). \tag{2.7}$$

Moreover, since the right-hand side above is symmetric with respect to the parameters a, b, we also have the continuous embedding

$$H^1_{b,a}\big(\mathbb{R}^N\big)\subset L^2_{(a+b+1)/2}\big(\mathbb{R}^N\big). \tag{2.8}$$

Next, we present some consequences of Theorem 2.1.

Corollary 2.3. *The inequalities below hold true with sharp constants*:

(i) *For any $u\in D^{1,2}(\mathbb{R}^N)$ it follows that*

$$\left(\frac{N-2}{2}\right)^2\int\limits_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}\,dx\leqslant\int\limits_{\mathbb{R}^N}|\nabla u|^2\,dx;$$

(ii) *For any $u\in H^1_{b+1,b}(\mathbb{R}^N)$ it follows that*

$$\left(\frac{N-2(b+1)}{2}\right)^2\int\limits_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2(b+1)}}\,dx\leqslant\int\limits_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2b}}\,dx;$$

(iii) *For any $u\in H^1_{a,a+1}(\mathbb{R}^N)$ it follows that*

$$\left(\frac{N-2(a+1)}{2}\right)^2\int\limits_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2(a+1)}}\,dx\leqslant\left(\int\limits_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}\,dx\right)^{1/2}\left(\int\limits_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2(a+1)}}\,dx\right)^{1/2};$$

(iv) *If $u\in H^1_{-(b+1),b}(\mathbb{R}^N)$ then $u\in L^2(\mathbb{R}^N)$ and*

$$\left(\frac{N}{2}\right)\int\limits_{\mathbb{R}^N}|u|^2\,dx\leqslant\left(\int\limits_{\mathbb{R}^N}|x|^{2(b+1)}|u|^2\,dx\right)^{1/2}\left(\int\limits_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2b}}\,dx\right)^{1/2};$$

(v) *If $u\in H^1_{0,1}(\mathbb{R}^N)$, $N>2$, then $u\in L^2_1(\mathbb{R}^N)$ and*

$$\left|\frac{N-2}{2}\right|\int\limits_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}\,dx\leqslant\left(\int\limits_{\mathbb{R}^N}|u|^2\,dx\right)^{1/2}\left(\int\limits_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^2}\,dx\right)^{1/2};$$

(vi) *If $u\in H^1_{-1,1}(\mathbb{R}^N)$, $N>1$, then $u\in L^2_{1/2}(\mathbb{R}^N)$ and*

$$\left(\frac{N-1}{2}\right)\int\limits_{\mathbb{R}^N}\frac{|u|^2}{|x|}\,dx\leqslant\left(\int\limits_{\mathbb{R}^N}|x|^2|u|^2\,dx\right)^{1/2}\left(\int\limits_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^2}\,dx\right)^{1/2};$$

[1] Note that the left-hand side of (2.4) vanishes when $\hat{C}=\hat{C}(a,b)=0$, that is $a+b+1=N$.

(vii) *If* $u \in H^1(\mathbb{R}^N) = H^1_{0,0}(\mathbb{R}^N)$, $N > 1$, *then* $u \in L^2_{1/2}(\mathbb{R}^N)$ *and*

$$\left(\frac{N-1}{2}\right) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|}\,dx \leqslant \left(\int_{\mathbb{R}^N} |u|^2\,dx\right)^{1/2} \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{1/2}.$$

Proof. We make special choices of a, b in (2.4) as follows:

(i) Let $a = 1, b = 0$;
(ii) Let $a = b + 1$;
(iii) Let $b = a + 1$;
(iv) Let $a = -b - 1$;
(v) Let $a = 0, b = 1$;
(vi) Let $a = -1, b = 1$;
(vii) Let $a = 0, b = 0$.

From Theorem 2.1 we know that all the above constants are sharp. □

Remark 2.4. In the above corollary, item (i) is Hardy's inequality and (ii) is an extension of (i) (since (i) follows from (ii) with $b = 0$). Also note that (ii) and (iii) correspond to (2.4) with $a = b + 1$ and $b = a + 1$, respectively. Finally, we point out that, when $a = b + 1 < N/2$ in (2.4), the best constant $\hat{C}(b + 1, b) = [\frac{N}{2} - (b + 1)]$ (see (ii) and (i)) is not achieved, as shown by Chou and Chu in [5]. We provide below a different proof of this result as a consequence of our approach in Theorem 2.1. In fact, we show that this noncompactness result holds true for any $a = b + 1 \neq \frac{N}{2}$ in (2.4).

Corollary 2.5. *Suppose* $a = b + 1 \neq \frac{N}{2}$ *in* (2.4). *Then, the best constant* $\hat{C}(b + 1, b) = |\frac{N}{2} - (b + 1)|$ *is never achieved, that is, for all* $u \in H^1_{a,b}(\mathbb{R}^N)$, $u \neq 0$, *one has*

$$\frac{\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}\,dx}{\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2(b+1)}}\,dx} > \left[\frac{N}{2} - (b+1)\right]^2.$$

Proof. From the proof of Theorem 2.1 we note that (2.5) and (2.4) are equivalent with $u \in H^1_{a,b}(\mathbb{R}^N)$. In addition, equality holds in (2.4) if and only if we have equality in (2.5), that is, if and only if one has

$$\frac{\nabla u}{|x|^b} + t\frac{x}{|x|^{a+1}}u = 0 \quad \text{a.e. in } \mathbb{R}^N \tag{2.9}$$

for some $t \in \mathbb{R}$. In this case we have $t = \frac{B}{2A}$, and it follows that

$$\operatorname{sgn}(t) = \operatorname{sgn}\big(N - (a + b + 1)\big). \tag{2.10}$$

On the other hand, an easy integration of the differential equation (2.9) yields (for arbitrary $C \in \mathbb{R}$)

$$u(x) = Ce^{-\frac{t}{\beta}|x|^\beta} \quad \text{when } \beta := b - a + 1 \neq 0, \tag{2.11}$$

or

$$u(x) = C\frac{1}{|x|^t} \quad \text{when } \beta := b - a + 1 = 0. \tag{2.12}$$

Now, let us consider the situation (2.12) in which $\beta = 0$, that is, $a = b + 1$. (Note that, in view of (2.10), we have $t > 0$ when $b + 1 < \frac{N}{2}$, whereas $t < 0$ when $b + 1 > \frac{N}{2}$.) Since $|x|^\gamma \notin L^2_{b+1}(\mathbb{R}^N)$ for any $\gamma \in \mathbb{R}$, it is clear that the functions in (2.12) do not belong to $H^1_{b+1,b}(\mathbb{R}^N)$ if $C \neq 0$. Therefore, the best constant in (2.4) is never achieved in this case. □

In fact, the above proof also allows us to consider the case $\beta := b - a + 1 \neq 0$ and to get information on whether the best constant $\hat{C}(a,b) = \frac{|N-(a+b+1)|}{2}$ in (2.4) is attained or not. We have the following:

Corollary 2.6. *Suppose $\beta := b - a + 1 \neq 0$ in* (2.4).

(i) *If $a < b + 1 < \frac{N}{2}$ or if $a > b + 1 > \frac{N}{2}$, then any u in* (2.11) *with $C \neq 0$ is a minimizer in $H^1_{a,b}$ for $\hat{C}(a,b)$*;
(ii) *If $b + 1 > a > \frac{N}{2}$ or if $b + 1 < a < \frac{N}{2}$, then no minimizer in $H^1_{a,b}$ exists for $\hat{C}(a,b)$.*

Proof. It suffices to note that if case (i) holds, then we either have

$$N - 2(b+1) < N - (a+b+1) < N - 2a,$$

which implies $\beta > 0$ and $t > 0$ (in view of (2.10)), or else we have

$$N - 2a < N - (a+b+1) < N - 2(b+1),$$

which implies $\beta < 0$ and $t < 0$ (again in view of (2.10)). In either situation, it follows that $\frac{t}{\beta} > 0$ and it can be easily checked that the nonzero functions $u(x)$ in (2.11) belong to $H^1_{a,b}$ and, therefore, are minimizers for $\hat{C}(a,b)$.

On the other hand, if case (ii) holds, then we either obtain $\beta > 0$ and $t < 0$, or $\beta < 0$ and $t > 0$, so that the candidates for minimizers given in (2.11) do not belong to $H^1_{a,b}$. □

As a final consequence of Theorem 2.1, we have the following:

Corollary 2.7. *Let $\hat{K}(a,b)$ and $\hat{K}(b,a)$ (with $a + b + 1 \neq N$) be the best constants in the continuous embedding $H^1_{a,b}(\mathbb{R}^N) \subset L^2_{(a+b+1)/2}(\mathbb{R}^N)$ and $H^1_{b,a}(\mathbb{R}^N) \subset L^2_{(a+b+1)/2}(\mathbb{R}^N)$, respectively, that is,*

$$\hat{K}(a,b) = \inf_{0 \neq u \in H^1_{a,b}} \frac{\|u\|^2_{H^1_{a,b}}}{\|u\|^2_{L^2_{(a+b+1)/2}}}, \qquad \hat{K}(b,a) = \inf_{0 \neq u \in H^1_{b,a}} \frac{\|u\|^2_{H^1_{b,a}}}{\|u\|^2_{L^2_{(a+b+1)/2}}}.$$

Then, we have $\hat{K}(a,b)$, $\hat{K}(b,a) \geqslant |N - (a+b+1)|$ and, if either $b + 1 > a > \frac{N}{2}$ or $b + 1 < a < \frac{N}{2}$ (i.e., in case (ii) *of Corollary* 2.6*), it follows that neither $\hat{K}(a,b)$ nor $\hat{K}(b,a)$ is achieved. In particular, the embeddings in* (2.7) *and* (2.8) *in those cases are not compact.*

Proof. From (2.4) and the elementary inequality $AB \leqslant \frac{1}{2}(A^2 + B^2)$, we have

$$\hat{C}(a,b)\|u\|^2_{L^2_{(a+b+1)/2}} \leqslant \|u\|_{L^2_a}\|u\|_{D^{1,2}_b} \leqslant \frac{1}{2}\big(\|u\|^2_{L^2_a} + \|u\|^2_{D^{1,2}_b}\big) := \frac{1}{2}\|u\|^2_{H^1_{a,b}},$$

as well as a similar expression with a and b exchanged. From the above estimate and the fact that $\hat{C}(a,b)=\hat{C}(b,a)=\frac{|N-(a+b+1)|}{2}:=\hat{C}$, it readily follows that both $\hat{K}(a,b)$ and $\hat{K}(b,a)$ are greater than or equal to $2\hat{C}=|N-(a+b+1)|$. Moreover, since $\hat{C}$ is not attained in case (ii) of Corollary 2.6, we conclude that neither $\hat{K}(a,b)$ nor $\hat{K}(b,a)$ can be achieved in that case. $\square$

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